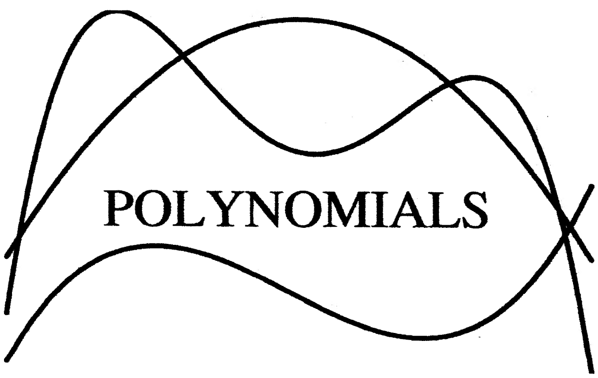
Project *MATHEMATICS!*

Program Guide and Workbook

to accompany the videotape on



Project MATHEMATICS!
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, CA 91125 • (818) 356-3759

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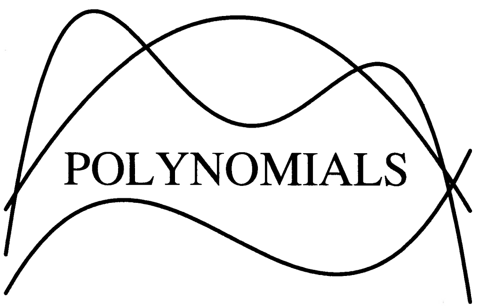

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AIMS AND GOALS OF *Project MATHEMATICS!*

Project MATHEMATICS! produces computer-animated videotapes to show students that learning mathematics can be exciting and intellectually rewarding. The videotapes treat mathematical concepts in ways that cannot be done at the chalkboard or in a textbook. They provide an audiovisual resource to be used together with textbooks and classroom instruction. Each videotape is accompanied by a workbook designed to help instructors integrate the videotape with traditional classroom activities. Video makes it possible to transmit a large amount of information in a relatively short time. Consequently, it is not expected that all students will understand and absorb all the information in one viewing. The viewer is encouraged to take advantage of video technology that makes it possible to stop the tape and repeat portions as needed.

The manner in which the videotape is used in the classroom will depend on the ability and background of the students and on the extent of teacher involvement. Some students will be able to watch the tape and learn much of the material without the help of an instructor. However, most students cannot learn mathematics by simply watching television any more than they can by simply listening to a classroom lecture or reading a textbook. For them, interaction with a teacher is essential to learning. The videotapes and workbooks are designed to stimulate discussion and encourage such interaction.

STRUCTURE OF THE WORKBOOK

The workbook begins with a brief outline of the video program, followed by suggestions of what the teacher can do before showing the tape. Numbered sections of the workbook correspond to capsule subdivisions in the tape. Each section summarizes important points in the capsule. Some sections contain exercises that can be used to strengthen understanding. The exercises emphasize key ideas, words and phrases, as well as applications. Some sections suggest projects that students can do for themselves.

I. BRIEF OUTLINE OF THE PROGRAM

The videotape begins with a brief introduction explaining that the program is about polynomials and the shape of their graphs. The program requires some familiarity with drawing graphs of polynomials in a rectangular coordinate system. To solve the exercises in the workbook, the student should know how to add or multiply polynomials.

The program opens with examples of polynomial curves that appear in real life. This is followed by a systematic description of polynomials by degree. Linear polynomials are discussed first; their graphs are straight lines of various slopes. Quadratic polynomials are discussed next; their graphs are parabolas, the prototype being the graph of $y = x^2$. It is shown how the Cartesian equation changes if the curve is translated vertically or horizontally, and also how the equation is altered by a vertical change of scale. The general quadratic polynomial is obtained from the prototype $y = x^2$ by a combination of horizontal translation, vertical scaling, and vertical translation. The next section deals with intersections of lines and parabolas and leads naturally to zeros of polynomials. Cubic polynomials are treated next, with discussion of possible zeros, local maxima, local minima, and points of inflection. There are basically three different cubic curves from which all cubics can be obtained by horizontal or vertical translation, or by horizontal or vertical change of scale, or by taking mirror images. A similar discussion is given for quartics and higher degree polynomials. The next section describes synthetic division as a method for calculating polynomials and for dividing a polynomial by a linear factor.

II. BEFORE WATCHING THE VIDEOTAPE

Students viewing the videotape should know how to manipulate polynomials algebraically and draw their graphs in a rectangular coordinate system. An effort should be made to review these skills before viewing the tape. Students should also be familiar with the key words and statements listed below and should be able to solve the exercises on page 6.

KEY WORDS AND STATEMENTS:

A *polynomial* in a single variable x is a function $f(x)$ that can be put in the form

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} + c_nx^n.$$

The numbers $c_0, c_1, \ldots, c_n$ are called *coefficients* of the polynomial. The coefficients can be real or complex numbers, but in this program only real coefficients are considered.

The number c_0 is called the *constant term* or the *constant coefficient*; c_1 is called the *linear coefficient*, and c_1x is the *linear term.*

The term c_nx^n containing the highest power of x with a nonzero coefficient is called the *leading term*; the nonnegative integer n is called the *degree* of the polynomial, and c_n is called the *leading coefficient.*

A polynomial of degree 0 is called a *constant polynomial.* The polynomial with all coefficients zero is called the *zero polynomial*; we do not assign it a degree. Polynomials of low degree are given special names:

degree:	1	2	3	4	5
name:	*linear*	*quadratic*	*cubic*	*quartic*	*quintic*

A *zero* of a polynomial $f(x)$ is any number r such that $f(r) = 0$. The zeros of a polynomial are real or complex numbers, but this program does not require knowledge of complex numbers.

THE MAIN IDEAS IN THIS PROGRAM:

This program discusses the shapes of graphs of polynomials with real coefficients.

The graphs of constant polynomials are horizontal straight lines.

The graphs of linear polynomials are nonhorizontal straight lines.

The graphs of all quadratic polynomials are parabolas obtained from the graph of $y = x^2$ by horizontal or vertical translation, horizontal or vertical change of scale, reflection, or by a combination of these.

The graphs of all cubic polynomials can be similarly obtained from the graphs of three basic cubics: $y = x^3$, $y = x^3 - x$, and $y = x^3 + x$.

The graphs of quartic polynomials can be obtained from the graphs of a basic set of polynomials, but the basic set is not finite. The same is true for polynomials of higher degree.

III. TABLE OF CONTENTS

Review of Prerequisites

This program discusses polynomials and the shape of their graphs in a rectangular coordinate system. We classify polynomials according to their degree. Polynomials of degree 0 are called *constant polynomials*; their graphs are horizontal straight lines as shown by the examples in Figure 1(a). Polynomials of degree 1 are called *linear polynomials*; examples of their graphs--nonhorizontal straight lines--are shown in Figure 1(b).

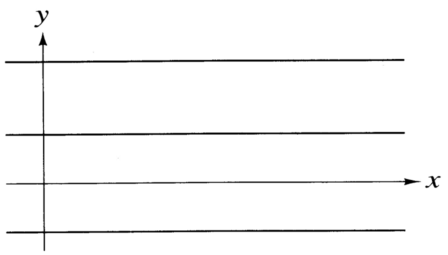


Figure 1(a). Graphs of constant polynomials.

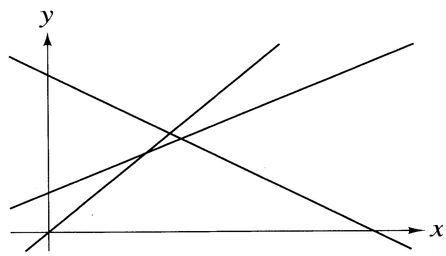


Figure 1(b). Graphs of linear polynomials.

Polynomials of degree 2 are called *quadratic polynomials*. Examples of their graphs, *parabolas*, are shown in Figure 2.

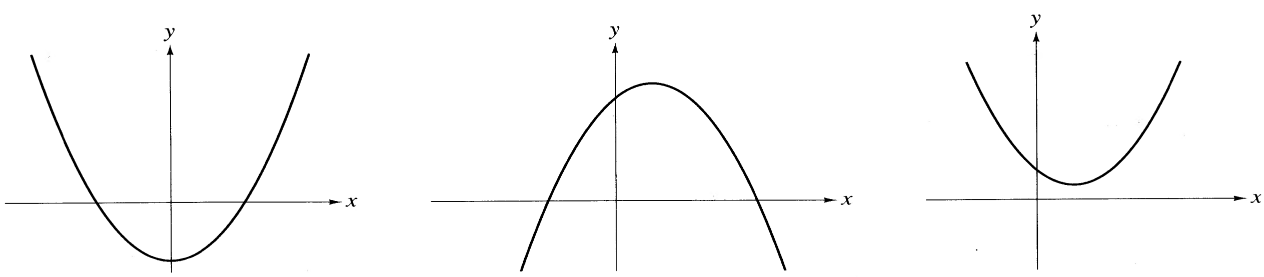


Figure 2. Graphs of quadratic polynomials.

Polynomials of degree 3 are called *cubic polynomials*, and their graphs are called *cubic curves*; examples are shown in Figure 3.

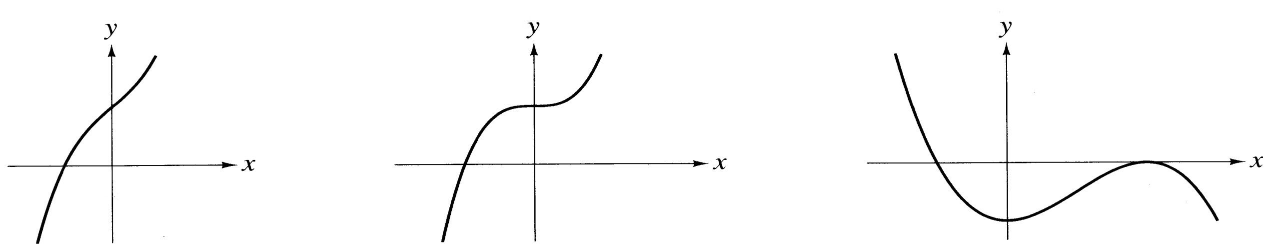


Figure 3. Graphs of cubic polynomials.

Polynomials of degree 4 are called *quartic polynomials*. Figure 4 shows examples of their graphs.

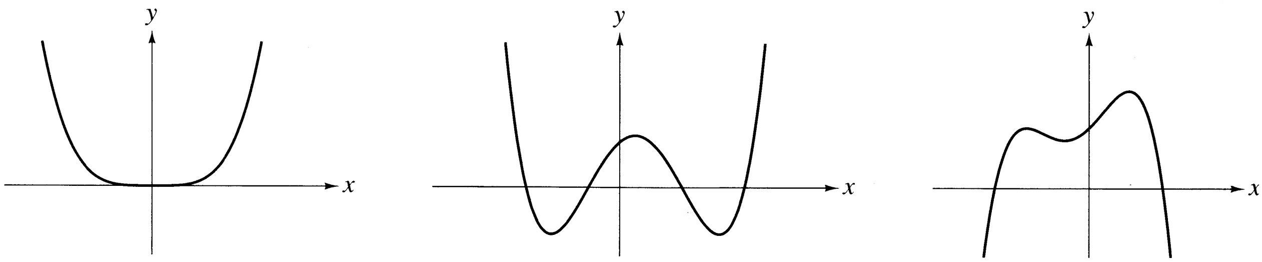


Figure 4. Graphs of quartic polynomials.

Figure 5 shows examples of graphs of *quintic polynomials* (degree 5).

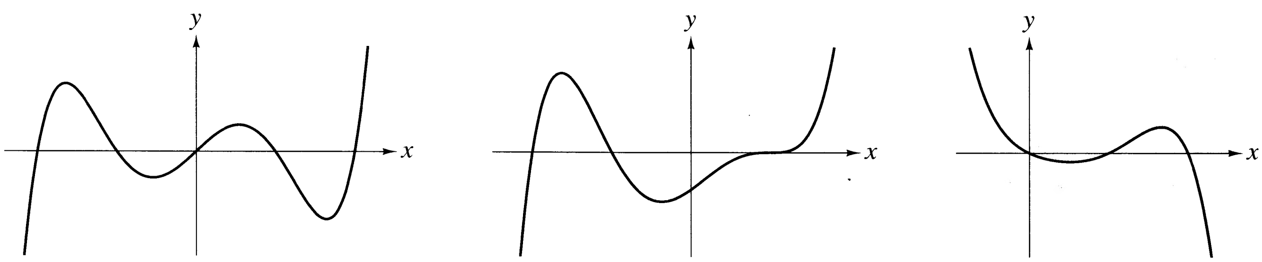
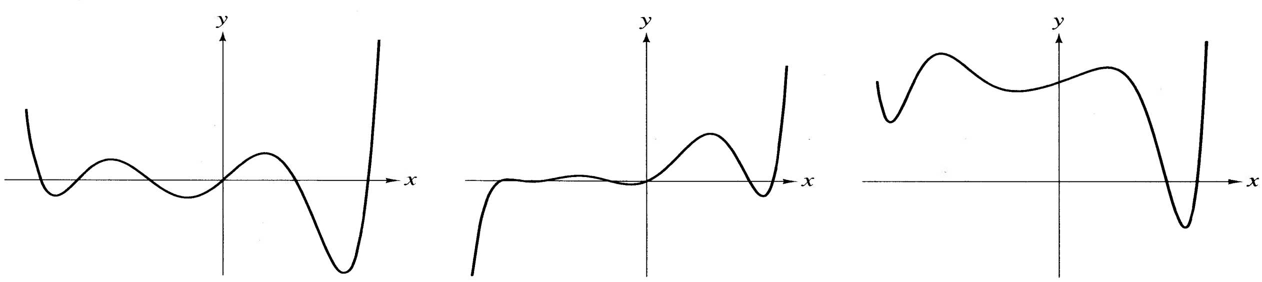


Figure 5. Graphs of quintic polynomials.

Examples of graphs of polynomials of degree 6, 7 and 8 are shown in Figure 6.



degree 6 degree 7 degree 8

Figure 6. Graphs of polynomials of higher degree.

EXERCISES

1. For each of the following polynomials, determine the degree, the constant term, the linear coefficient, and the leading coefficient.

(a) $5x^3 - 7x^2 + 9x + 4$ (b) $5x^3 - 7x^2 + 4$ (c) $x(5x^3 - 7x^2 + 9x + 4)$

(d) $(5x^3 - 7x^2 + 9x) + (5x^3 - 2x^2 + x - 2)$ (e) $(5x^3 - 7x^2 + 9x) - (5x^3 - 2x^2 + x - 2)$

2. Do the same as in Exercise 1 for each of the following polynomials. The problem can be solved without carrying out all the indicated multiplications.

(a) $x^2(5x - 7)$ (b) $(5x^2 - 7)(x + 3)$ (c) $x(x - 1)(x - 2)$

(d) $x^2(x^2 - 1)(x - 2)$ (e) $(x + 1)^2 - (x - 1)^2$ (f) $x^4(x + 1)^2 - x^2(x - 1)^4$

3. For each of the following statements, give a proof if you think the statement is always true, or exhibit a counter example if it is sometimes false.

(a) The sum of two polynomials of degree n is a polynomial of degree n.

(b) The degree of a product of two polynomials is the sum of the degrees of the two factors.

4. Show that each of the following functions is a polynomial, and find its degree, its constant term, and its leading coefficient.

(a) $(1 + x^2 + x^4 + x^6)/(1 + x^4)$ (b) $(1 + x^6)/(1 + x^2)$

5. Show that each of the following functions is *not* a polynomial.

(a) $(1 + x^6)/(1 + x^4)$ (b) $(1 + x^4)/(1 + x^2)$ (c) $1/(1 + x^2)$ (d) $(1 + x^2)^{1/2}$

6. (a) If two linear polynomials $c_0 + c_1x$ and $d_0 + d_1x$ take equal values for two distinct values of x, show that $c_0 = d_0$ and $c_1 = d_1$.

(b) State and prove a corresponding result for two quadratic polynomials.

7. Because two points determine a line, the graph of a linear polynomial is determined by specifying two points on it. Find the linear polynomial whose graph contains each of the following pairs of points, and draw the graph in each case.

(a) (1, 1) and (–1, –1) (b) (1, 2) and (2, 3) (c) (1, 0) and (2, 1)

8. In each case, find all s and t such that the following three points lie on a line:

(a) $(1, 2)$, $(2, s)$, $(3, t)$ (b) $(1, 3)$, $(2, 5)$, (s, t) (c) $(0, 0)$, $(s, 4)$, $(3t, 4t)$

9. If the graph of a linear polynomial intersects the x axis at $x = a$ and intersects the y axis at $y = b$, where neither a or b is zero, show that every point (x, y) on the line satisfies the equation $x/a + y/b = 1$.

1. *Polynomials in real life*

Note: The reader may wish to postpone studying this section until after reading sections 2 and 3.

The simplest polynomial graphs, straight lines, occur in one of the fundamental laws of classical physics, the *law of inertia*:

A body will remain at rest or continue to move at constant speed ***in a straight line*** *unless acted on by an outside agent.*

When a basketball is thrown toward a basket, the ball does *not* travel in a straight line because outside agents act on it. For example, air resistance slows it down, and the force of gravity pulls it back to earth. In 1638 the Italian Renaissance thinker Galileo Galilei showed that, in the absence of air resistance, the force of gravity will cause a ball, or any projectile, to move along a parabola. In a moment we'll see why. But first we'll show that a body travels along a straight line if its horizontal component of speed is constant and if its vertical component of speed is constant.

Choose the x axis along the horizontal direction of motion, and the y axis pointing vertically upward. Assume the projectile is launched from the origin and moves without external forces such as air resistance or gravity acting on it. Assume also that it has a constant horizontal component of speed v_1, and a constant vertical component of speed v_2. Because v_1 is constant, the horizontal distance x the projectile moves in time t is the product of horizontal speed times time:

$$x = v_1 t. \tag{1}$$

Similarly, the vertical distance y moved in time t is

$$y = v_2 t. \tag{2}$$

Note that both the horizontal distance x and the vertical distance y are linear polynomials in t. To show that y is also a linear polynomial in x we eliminate t from the two equations. From Equation (1) we find $t = x/v_1$. Replacing t by x/v_1 in Equation (2) we obtain

$$y = v_2(x/v_1) = ax,$$

where $a = v_2/v_1$. Thus y is a linear polynomial in x, so its graph is a straight line, as shown in Figure 7.

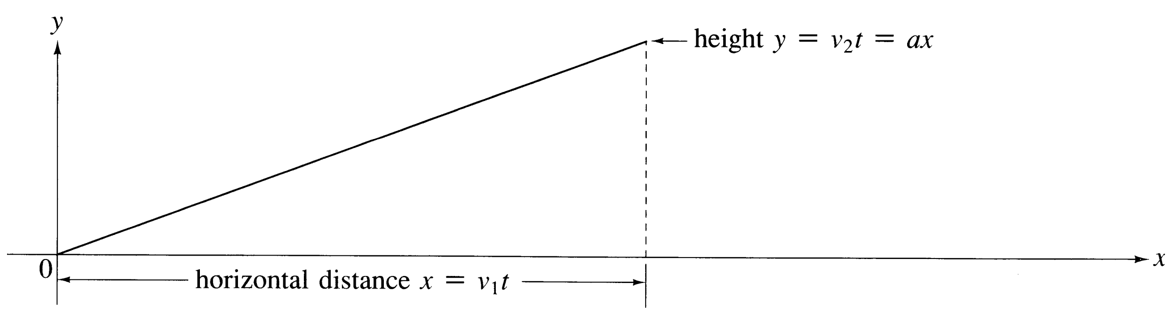


Figure 7. Linear motion of a projectile in the absence of gravity or air resistance.

In the presence of gravity, (but in the absence of air resistance or other external forces) Equation (1) for the horizontal distance x does not change, but Equation (2) for the vertical distance y must be altered by the distance a body falls in time t. Galileo observed that under the force of gravity a body falls vertically through a distance proportional to the square of time, that is, through a distance ct^2, where c is a constant, the same for all falling bodies. The constant c is numerically equal to the distance a body falls in one unit of time. If t is measured in seconds and y in feet, then c is approximately 16. (The value of c can be determined by experimenting with actual falling bodies. On another planet, c might have a different value, but the distance fallen will still be ct^2.) Therefore, in place of Equation (2), the projectile's actual height y at time t is given by

(3) $$y = v_2 t - ct^2.$$

Note that y is a quadratic polynomial in t, the quadratic term being due to gravity. To relate y to x, we again eliminate t. Replace t by x/v_1 from Equation (1) and use this in Equation (3) to find

(4) $$y = v_2(x/v_1) - c(x/v_1)^2 = ax - bx^2,$$

where $a = v_2/v_1$ and $b = c/(v_1)^2$. This shows that y is a quadratic polynomial in x, and the trajectory is indeed a parabola. Figure 8 shows the parabolic path and its deviation from the linear path of Figure 7.

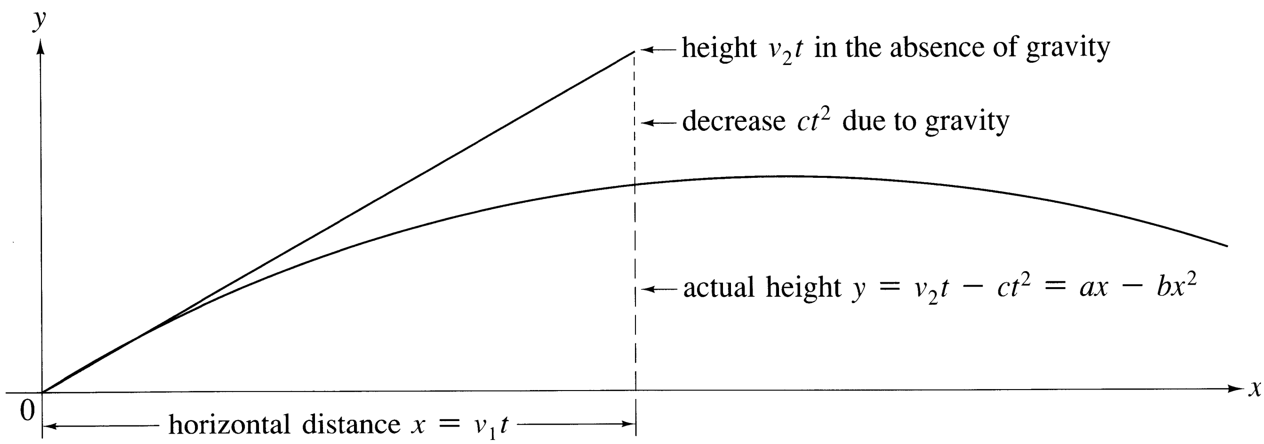


Figure 8. Parabolic motion of a projectile under the influence of gravity.

The parabola is one member of a family of curves called *conic sections* that can be formed by slicing a cone with a plane. We'll encounter other conic sections, the *ellipse* and *hyperbola*, in another program.

The study of conic sections originated in ancient Greece. When suffering citizens appealed to the Oracle of Delos to halt a plague that ravaged Athens, they were instructed to double the volume of Apollo's cubical altar. The Greeks attempted to do so using straightedge and compass, but their efforts were doomed to failure because, as we know now, the volume of a cube cannot be doubled in this way. Legend has it that the pestilence lasted until about 340 B.C., when Menaechmus found two solutions using conic sections, one by intersecting a parabola and hyperbola, the other by intersecting two parabolas.

Apollonius of Perga (262-200 B.C.) wrote the first comprehensive treatise on conic sections. For this eight-volume work, one of the most profound achievements of classical Greek geometry, he earned the title of "the great geometer."

Cubic splines

Cubic curves have long been used by loftsmen in laying out hulls of ships. The shapes are formed by a mechanical device, called a *spline,* which consists of a long narrow strip of wood or elastic material equipped with a groove and a set of lead weights (called *ducks*) with attached arms designed to fit in the groove. (See Figure 9a.) When the ducks are properly located along the rod, as shown in Figure 9b, the spline forms a smooth curve that passes through a number of prescribed points. In the mid-1700s Leonhard Euler and the Bernoullis showed that between any pair of junction points a thin elastic beam would, under small deflections, bend in a cubic curve. Therefore, the spline forms a smooth curve consisting of pieces of several cubic curves fitted together. These curves are called *cubic splines.*

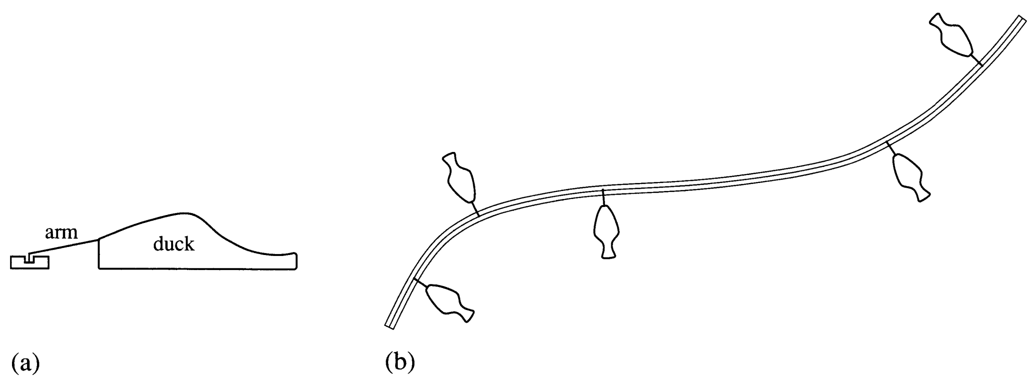


Figure 9. A mechanical spline forms a curve consisting of pieces of cubic curves fitted together.

A single polynomial of higher degree can also be used to pass a curve through a prescribed set of junction points. But as the number of junction points increases, the degree of the polynomial also increases and undesirable wiggles are sometimes introduced. The wiggles can be avoided by using cubic splines. This is illustrated in Figure 10, which shows five junction points (±2, 0), (±1, 0), and (0, 1) joined by a quartic curve (the dotted curve) and by a cubic spline (the solid curve) which consists of four cubic pieces.

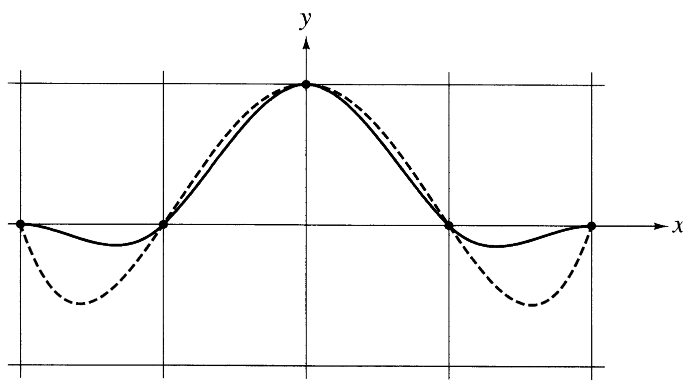


Figure 10. Five points joined by a quartic (the dotted curve) and by a cubic spline (the solid curve).

For ease in computation, low-degree polynomials are preferred. Linear pieces could produce undesirable sharp corners at the junction points. Parabolic pieces always bend up or down between a pair of junction points. Cubics are the polynomials of lowest degree with a point of inflection that permits a change in direction of bending between each pair of junction points.

EXERCISES

1. Refer to Figure 8, p. 8, which shows the parabolic trajectory of a projectile moving under the influence of gravity but in the absence of air resistance or other external forces. The equation of the curve is

$$y = ax - bx^2. \tag{4}$$

In Section 1 it was shown that a and b are positive numbers given by the formulas $a = v_2/v_1$ and $b = c/(v_1)^2$, where v_1 and v_2 are the horizontal and vertical components of the speed with which the projectile was thrown, and c is a constant due to gravity. This exercise shows how to find the horizontal distance the projectile travels before it returns to the ground, and the maximum height it attains during its flight.

(a) The equation for y can be rewritten as follows: $y = x(a - bx)$. If the product of two numbers is zero, at least one of the factors must be zero. Use this fact to show that the horizontal distance the projectile travels before it returns to the ground is a/b.

(b) To find the maximum value of y we rewrite the quadratic polynomial as a difference of two squares by a method called *completing the square*. First write the equation for y as follows:

$$y = -b(x^2 - \frac{a}{b}x).$$

The two terms in parentheses, $x^2 - \frac{a}{b}x$, occur in squaring the binomial $x - \frac{a}{2b}$. Use this fact to show that the equation for y can be rewritten in the form

$$y = \frac{a^2}{4b} - b(x - \frac{a}{2b})^2.$$

The term $a^2/(4b)$ on the right is positive and does not depend on x; the second term, which depends on x, is negative or zero. Therefore y will take its largest value when we subtract as little as possible from the first term. When $x = a/(2b)$ we subtract zero, so the maximum value of y is $a^2/(4b)$. The number $a/(2b)$ is half the horizontal distance the projectile travels before returning to the ground, so the projectile travels equal distances horizontally during ascent and descent.

Note. It should be remembered that these calculations ignore the effect of air resistance. Real projectiles in air do not move in exact parabolic trajectories, or travel equal horizontal distances in ascent and descent. For example, air resistance can reduce the distance a well-hit baseball travels by as much as 40%. (See P. Brancazio, *Scientific American*, Vol. 28 (April 1983), p. 76.) Real projectiles of high density, small cross-sectional area, and relatively low speed follow Galileo's parabolic trajectory fairly closely. A shot-put follows a parabolic orbit more closely than a basketball.

2. Use Equation (3), $y = v_2t - ct^2$, which expresses the height y in terms of time t, to determine the total time it takes the projectile to reach the ground. Find the time it takes for the projectile to reach its maximum height. What is the relation of this to the total time of flight?

3. In Figure 10, join consecutive junction points by line segments and compare with the cubic spline.

4. (a) Show that more than one parabolic arc passes through two given points.

(b) How would you join the junction points in Figure 10 with four parabolic pieces to avoid sharp corners at the junction points?

2. *Linear polynomials*

We say that y is a linear polynomial in x if

$$y = ax + b, \tag{5}$$

where the coefficients a and b are given numbers with $a \neq 0$. (The case $a = 0$ gives constant polynomials, whose graphs are horizontal straight lines.) The polynomial in (5) is called linear because its graph is a straight line. In fact, given any nonvertical straight line we will show that the rectangular coordinates x and y of any point on the line satisfy an equation of the form (5), with $a \neq 0$ if the line is not horizontal.

To do this we use the fact that one and only one straight line passes through two given points. For the given points choose (x, y) and (p, q), as shown in Figure 11, so that they are vertices of a right triangle whose legs are parallel to the coordinate axes and whose hypotenuse passes through the two points.

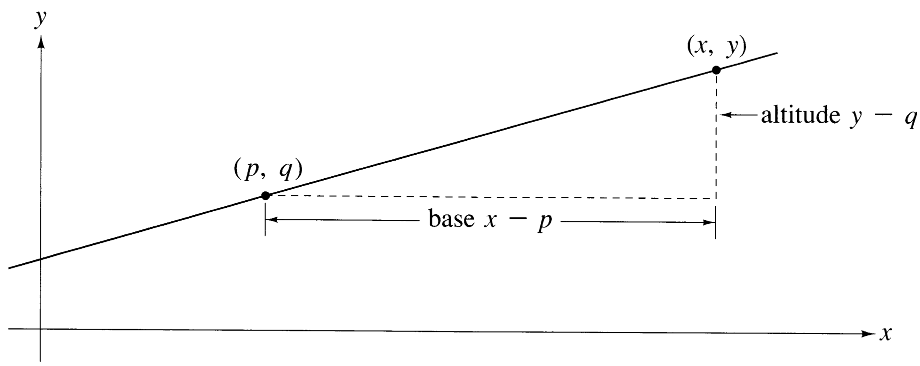


Figure 11. The ratio of altitude to base is constant as (x, y) varies along the line.

If (p, q) is kept fixed and (x, y) is allowed to vary along the line, the size of the right triangle may change, but all the right triangles so formed will be similar, so lengths of corresponding sides will have a constant ratio. In particular, the ratio of the altitude $y - q$ to the base $x - p$ is constant (that is, independent of x and y). If we call that constant m (the usual notation for this constant ratio) we have

$$\frac{y - q}{x - p} = m. \tag{6}$$

To solve Equation (6) for y we first clear the fractions to obtain $y - q = m(x - p)$, and we find

$$y = mx + b, \tag{7}$$

where $b = q - mp$. Thus y is indeed a linear polynomial in x. The constant term b is the value of the polynomial when $x = 0$, so it tells us where the graph crosses the y axis. For this reason, b is called the *y intercept* of the graph. When $b = 0$ the graph passes through the origin. The linear coefficient m is called the *slope* of the line. Its geometric meaning is revealed by Equation (6). The denominator, $x - p$, measures the change in horizontal distance between the two points (x, y) and (p, q), while the numerator, $y - q$, measures the corresponding change in vertical height; m is the ratio of these changes:

$$m = \text{slope} = \frac{\text{change in height}}{\text{change in horizontal distance}}. \tag{8}$$

When the change in horizontal distance is equal to 1, the slope is equal to the corresponding change in height. Therefore slope represents the change in height per unit of horizontal distance. The slope is positive if the height increases, negative if the height decreases, and zero if the height doesn't change.

Figure 12 shows lines with different slopes but with the same y-intercept. A horizontal line has slope zero and its equation is $y = b$. A line of slope 1 has equation $y = x + b$. A line of slope 2 is twice as steep, and its equation is $y = 2x + b$. The equation $y = -x + b$ describes a line with slope -1.

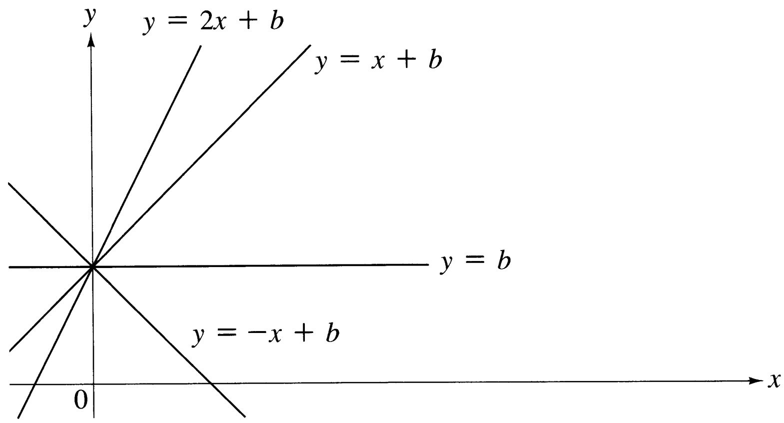


Figure 12. Lines of different slopes passing through the same point $(0, b)$.

On a vertical line every point has the same x coordinate, so the denominator in Eq. (8) is zero and slope is not defined. If the x coordinate has the constant value c, each point on the line satisfies the equation

$$x = c. \tag{9}$$

This is analogous to the equation $y = b$, which describes a horizontal line. All lines in the xy plane, vertical or not, can be described by an equation of the form

$$Ax + By = C, \tag{10}$$

where A, B, C are constants, and not both A and B are zero. If $B \neq 0$ the slope of the line is $-A/B$. If $B = 0$ the line is vertical.

Figure 13 shows two lines, $y = x - 1$ and $y = -\frac{1}{2}x + 2$, with different slopes. To find the point of intersection we equate the right members of the equations and solve for x. After a little algebra we obtain $x = 2$. From either linear equation we find $y = 1$ when $x = 2$, so the point of intersection is $(2, 1)$.

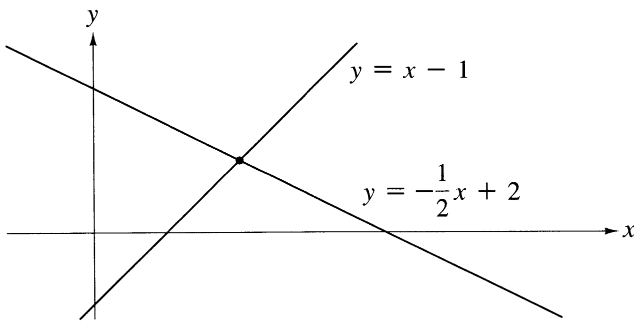


Figure 13. Two lines with different slopes intersect at a point.

In the same manner we can find the point of intersection of any two lines with different slopes, say $y = m_1x + b_1$ and $y = m_2x + b_2$. When we equate the right members we obtain $m_1x + b_1 = m_2x + b_2$, hence

$$(m_1 - m_2)x = b_2 - b_1. \tag{11}$$

If the slopes are different, $m_1 - m_2$ is not zero and there is a unique solution for x.

What if the slopes are equal? If $m_1 = m_2$ and $b_1 = b_2$ the linear equations are identical and represent the same line. But if $m_1 = m_2$ and $b_1 \neq b_2$, Equation (11) becomes $0 = b_2 - b_1$, contradicting the assumption $b_1 \neq b_2$; in this case no x satisfies (11), hence no pair (x, y) satisfies both equations. Therefore, two lines with equal slopes are either coincident (identical lines) or parallel (they never meet).

How to obtain the graph of any linear polynomial from that of y = x

A linear polynomial $y = ax + b$ contains two coefficients a and b, where $a \neq 0$. As a and b are allowed to vary, the geometric shape of the graph is always a straight line. The slope of the line is determined by a, and its location in the plane is determined by b, its y intercept. It is easy to show that all these lines can be obtained from the graph of $y = x$ by performing simple geometric operations.

Because $a \neq 0$, the equation $y = ax + b$ can be rewritten as follows:

$$y = a(x + \frac{b}{a}). \tag{12}$$

This equation can be obtained from the basic equation $y = x$ in two steps. First, replace x by $x + b/a$, then multiply the result by a. The first step transforms $y = x$ to the equation

$$y = x + \frac{b}{a} \ , \tag{13}$$

whose graph, shown in Figure 14b, is the line of slope 1 with y intercept b/a. Geometrically, this corresponds to vertical translation of the line $y = x$ by an amount b/a without changing its slope. (Or, equivalently, to a horizontal translation by an amount $-b/a$.) The second step, multiplication by a, represents a change of scale in the vertical direction by the scaling factor a if $a > 0$, or a change of scale together with a reflection through the x axis if $a < 0$; this changes Eq. (13) to $y = ax + b$. In other words, the graph of any linear polynomial can be obtained from $y = x$ by a horizontal or vertical translation followed by a change of scale in the vertical direction, with a possible reflection in the x axis if $a < 0$.

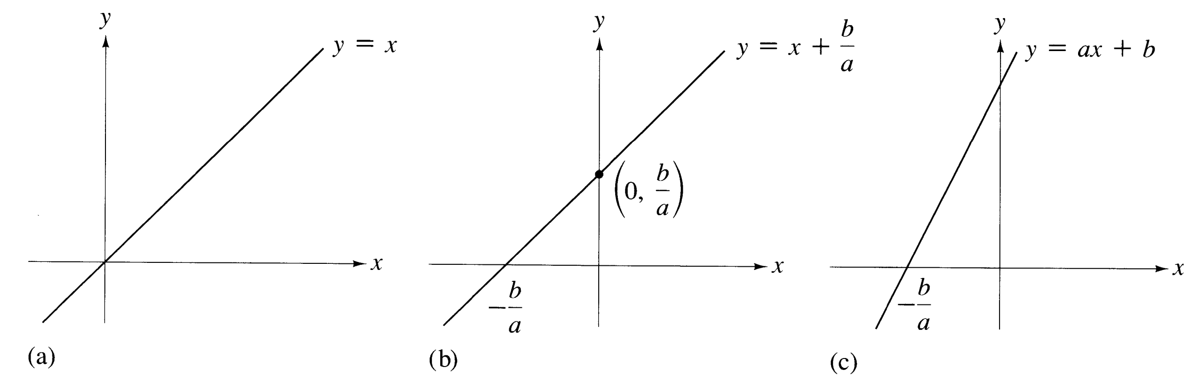


Figure 14. Lines obtained from $y = x$ by vertical translation (b) plus a vertical change of scale (c).

The same result can be obtained in a different manner. Start with the line $y = x$ in Figure 15a, and replace x by ax. If $a > 0$ all vertical distances on the line are multiplied by the factor a. Geometrically, this corresponds to a change of scale in the vertical direction by the scaling factor a; in particular, the slope of the line changes from from 1 to a. (See Fig. 15b.) If $a < 0$ this is a change of scale by the positive factor $-a$ together with a reflection through the x axis. To go from the equation $y = ax$ to the equation $y = ax + b$ we simply add b to all the vertical distances. (See Fig. 15c). This is the same as shifting the graph vertically by an amount b, upward if b is positive, downward if b is negative. So we see that a general linear polynomial can be obtained from the line $y = x$ by performing simple geometric operations: a change of scale in the vertical direction (with a possible reflection through the x axis) followed by a vertical translation.

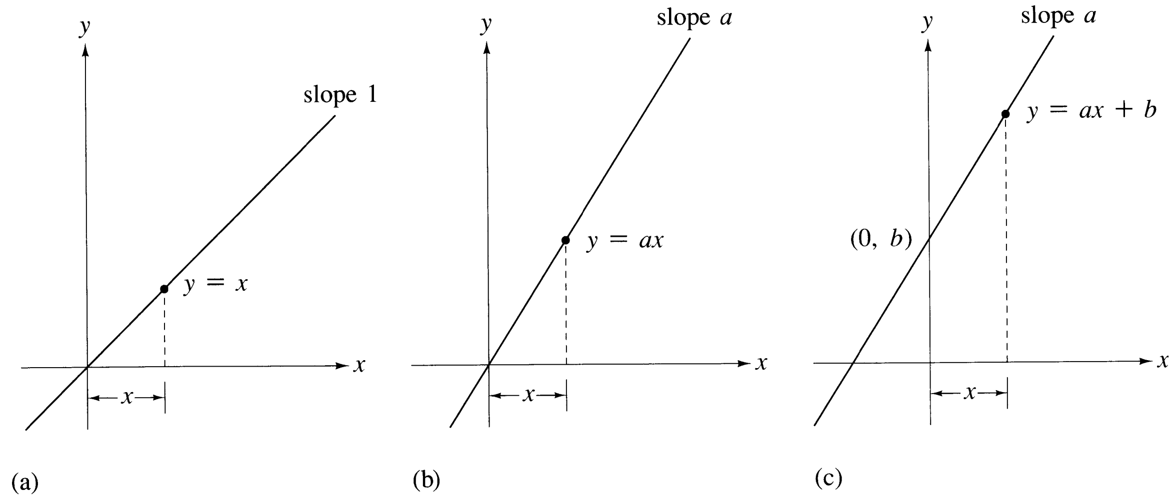


Figure 15. Lines obtained from $y = x$ by a vertical change of scale (b) and a vertical translation (c).

EXERCISES

1. Draw graphs of the lines described by the following equations,

(a) $y = x$, $y = x + 1$, $y = x - 1$. (b) $y = 2x$, $y = 2x + 2$, $y = -2x + 2$.

(c) $2x - 3y = 12$, $4x - 6y = 5$, $3x + 2y = 1$. (d) $x = 1$, $x = 2$, $x = 0$.

2. Find an equation of the form $y = mx + b$ for each line with the following properties:

(a) Slope 2, passing through the origin. (b) Slope -2, passing through $(0, 4)$.

3. Find an equation $y = mx + b$ for the line passing through each pair of points.

(a) $(0, 0)$ and $(6, 6)$. (b) $(0, 0)$ and $(3, 6)$. (c) $(0, 1)$ and $(3, 6)$.

4. A linear equation relates temperature F in degrees Fahrenheit with temperature C in degrees Celsius. It is given that $F = 32$ when $C = 0$ (the freezing point of water) and that $F = 212$ when $C = 100$ (the boiling point of water). Express F in terms of C, and C in terms of F.

5. The natural length of a spring is 8 inches and a force of 20 lbs is required for each inch the spring is increased in length. If a force of F lbs stretches the spring to length L inches, which of the following linear equations states these facts? (a) $F = 20L$. (b) $F = 20L + 8$. (c) $F = 20(L - 8)$.

6. For each of the following pairs of lines, determine if they are coincident, parallel, or intersect at one point. If they intersect at one point, determine the coordinates of this point.

(a) $y = 2x - 5,\ \ y = 2x + 3$. (b) $y = 3x - 2,\ \ y = 4x - 4$. (c) $y = \frac{1}{2}x - 1,\ \ 2y = x - 2$.

(d) $y = 5x + 1,\ \ y = -x$. (e) $y = x - 2,\ \ y = 4$. (f) $y = \dfrac{x+2}{2},\ \ y = \dfrac{x+2}{4}$.

7. A satellite makes a circular orbit around the earth at a constant altitude. What change in altitude is required to lengthen each orbit by one mile? (The circumference of a circle of radius r is $2\pi r$.)

8. Two cars move in opposite directions at constant speeds from a given point, one traveling 12 miles an hour faster than the other. After 4 hours they are 392 miles apart. Find the speed of each car.

9. A rope 10,920 kilometers long fits snugly around the moon's equator. The rope is lengthened by 1 meter to form a larger circle around the equator. How high will the new circle be above the surface?

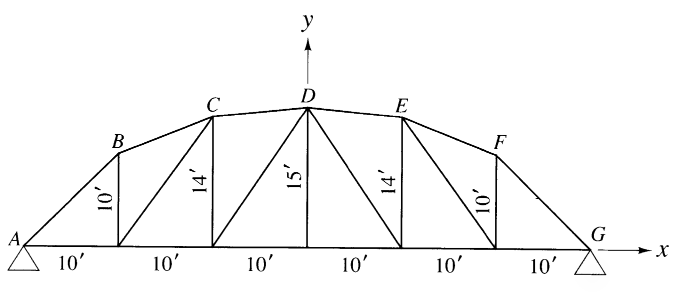
10. The linear polynomial $24x + 1$ is the square of an integer when x takes the three consecutive integer values 0, 1 and 2. The same is true for the linear polynomial $120x + 49$. Find a linear polynomial that is the square of an integer when x takes the three consecutive integer values 3, 4, and 5. There are infinitely many such polynomials. (It can be shown that there is *no* linear polynomial that is the square of an integer for *four* consecutive integer values of x.)

11. Three positive integers p, q, r are said to form a Pythagorean triple if $p^2 + q^2 = r^2$. For a given Pythagorean triple p, q, r, find integers a and b (expressed in terms of p, q, r) such that the polynomial $f(x) = ax + b$ takes the values $(p - q)^2$, r^2, and $(p + q)^2$ for three consecutive integer values of x.

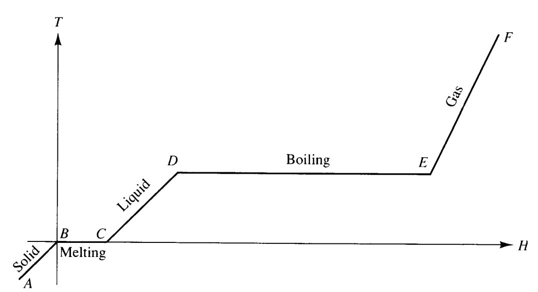
12. A bridge truss is shown in the diagram below. Determine a linear equation of the form $y = mx + b$ for each of the line segments AB, BC, CD, DE, EF, and FG.

13. In the graph of a water cycle shown below, T is the temperature in degrees Celcius and H is the heat quantity in calories per unit mass. Determine a linear equation for each line segment, given that

$A = (-20, -20)$, $B = (0,0)$, $C = (70, 0)$, $D = (170, 100)$, $E = (529, 100)$, $F = (625, 300)$.



Exercise 12. Bridge truss



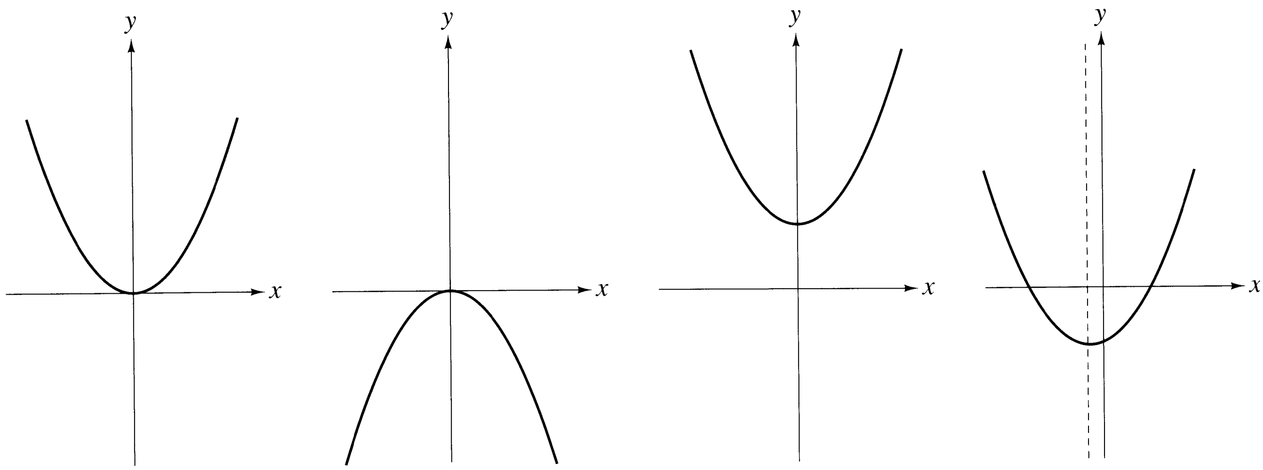
Exercise 13. Water cycle

3. *Quadratic polynomials*

Quadratic polynomials were encountered in our discussion of trajectories of projectiles. Recall that y is a quadratic polynomial in x if

$$y = ax^2 + bx + c, \tag{12}$$

where a, b and c are constants, with $a \neq 0$. Its graph is called a parabola. As the coefficients a, b, c vary independently, the shape and position of the parabola may change, but all the graphs will have certain features in common, as illustrated by the examples in Figure 16.



(a) $y = x^2$ (b) $y = -x^2$ (c) $y = x^2 + 1$ (d) $y = 2x^2 + x - 1$

Figure 16. The graph of a quadratic polynomial is a parabola.

The parabolas in (a), (c) and (d) open upward because the coefficient of x^2 is positive, while that in (b) opens downward because the coefficient of x^2 is negative.

All parabolas have an axis of symmetry. The curves in (a), (b) and (c) are symmetric about the y axis because their equations are unchanged if the point (x, y) is replaced by the symmetric point $(-x, y)$. The axis of symmetry of the curve in (d) is the line $x = -1/4$.

The point where a parabola crosses its axis of symmetry is called the *vertex* of the parabola. It is the lowest point (called a *minimum*) of curves (a), (c) and (d), and the highest point (called a *maximum*) of curve (b). In (a) and (b) the vertex is at the origin, where the curve touches the x axis.

A value of x that makes a polynomial equal to zero is called a *zero* of the polynomial. A quadratic polynomial whose graph crosses the x axis twice, like that in (d), has two distinct real zeros. They can be found by solving the quadratic equation $2x^2 + x - 1 = 0$. The left member can be factored to give

$$(2x - 1)(x + 1) = 0,$$

from which we see that the zeros are at $x = -1$ and $x = 1/2$. In examples (a) and (b) the quadratic equation for the zeros is $x^2 = 0$. In these examples the polynomial is said to have a *double zero* at $x = 0$. In example (c) the equation $x^2 + 1 = 0$ has no real solutions and the graph does not intersect the x axis.

4. *Intersections of lines and parabolas*

Many problems require finding the intersection points of two polynomial curves. If a point (x, y) is on two curves with equations $y = f(x)$ and $y = g(x)$, its coordinates satisfy both equations, so the x coordinate of every intersection point satisfies the equation $f(x) - g(x) = 0$. Therefore x is a zero of the polynomial $f(x) - g(x)$. If one polynomial is linear and the other is quadratic, the difference $f(x) - g(x)$ is quadratic, and the problem reduces to finding the zeros of a quadratic polynomial. For example, to find the intersection points of the parabola $y = x^2 + 1$ with a line through the origin, say the line $y = mx$ of slope m, we have to find the zeros of the quadratic polynomial $x^2 + 1 - mx$. The examples in Figure 17 show that the number of intersection points can be 0, 1, or 2. The line $y = x$ does not intersect the parabola; the line $y = 2x$ intersects it once; and the line $y = -3x$ intersects it twice.

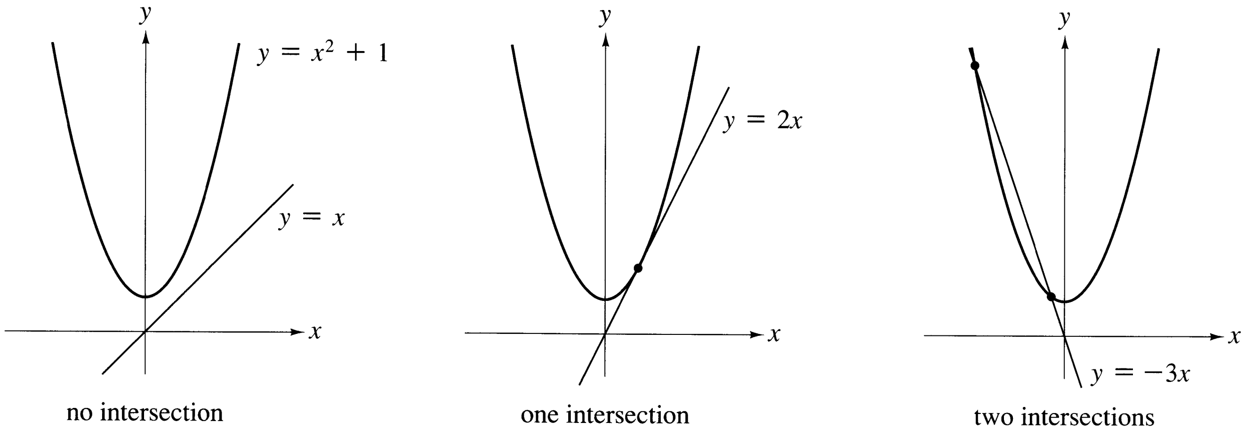


Figure 17. The number of intersection points of a parabola and a line can be 0, 1 or 2.

Determining the zeros of a quadratic polynomial

The zeros of a general quadratic polynomial $y = ax^2 + bx + c$ can be found by solving the quadratic equation $ax^2 + bx + c = 0$. If the quadratic can be factored as a product of two real linear polynomials, the zeros can be found by setting each factor equal to zero. The solutions can also be found from the following formula, called the *quadratic formula*, which we will derive in a moment:

$$x = \frac{-b \pm \sqrt{D}}{2a}, \quad \text{where } D = b^2 - 4ac. \tag{13}$$

The number D is called the *discriminant* of the quadratic polynomial; it determines the nature of the zeros. If the discriminant D is positive, formula (13) provides two distinct real zeros and the graph crosses the axis twice. If the discriminant D is zero, the quadratic is said to have a double zero and its graph touches the x axis at the point where $x = -b/(2a)$. But if the discriminant D is negative, there are no real zeros, because D has no real square roots, and the graph does not intersect the x axis. In this case there are two complex zeros.

Note. Although a knowledge of complex numbers is not assumed in this program, the reader should realize that a quadratic polynomial with no real zeros does have *complex* zeros. For example, the zeros of the quadratic polynomial $x^2 + 1$ are $\pm i$, where i is a complex number whose square is -1. Complex numbers were invented so that every quadratic polynomial would have at least one zero, real or complex.

EXAMPLE. To find the points of intersection of the parabola $y = x^2 + 1$ with a line $y = mx$ through the origin we need to solve the quadratic equation $x^2 - mx + 1 = 0$. The solutions given by the quadratic formula (13) are

$$x = \frac{m \pm \sqrt{D}}{2},$$

where $D = m^2 - 4$ is the discriminant. If $m^2 < 4$ the discriminant is negative and there are no real solutions. Therefore lines with slope between -2 and $+2$ will not intersect the parabola. If $m^2 = 4$ the discriminant is zero and there is one solution. In other words, each of the two lines $y = 2x$ and $y = -2x$ intersect the parabola at exactly one point. Finally, if $m^2 > 4$ the discriminant is positive and there are two solutions. So a line with positive slope $m > 2$ or negative slope $m < -2$ will intersect the parabola at exactly two points. The three cases are illustrated in Figure 17 on page 17.

How to obtain the graph of any quadratic polynomial from that of $y = x^2$

We can always obtain the graph of any quadratic polynomial $y = ax^2 + bx + c$ (for example, the one shown in Figure 18d) from that of $y = x^2$ (shown in Figure 18a) by the following geometric method. First, perform a horizontal translation to get the graph in Figure 18b, then a change of scale in the vertical direction to get the graph in Figure 18c, and finally a vertical translation to obtain the curve in Figure 18d. The difficult part is to figure out the amount of horizontal and vertical translation that is needed.

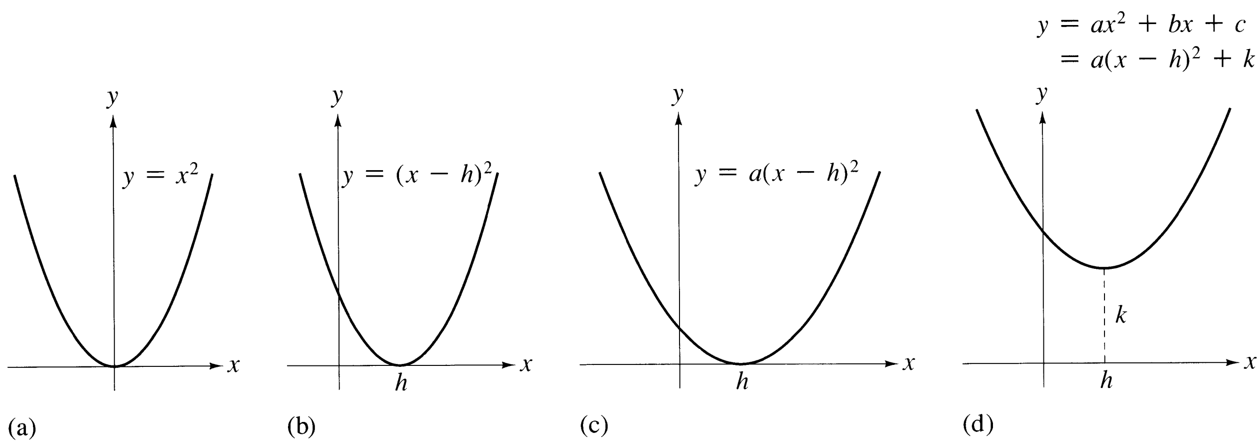


Figure 18. Graph of a general quadratic obtained from the special case $y = x^2$.

Horizontal translation by an arbitrary amount h can be achieved by replacing x by $x - h$ in the equation $y = x^2$. This gives us the equation

$$y = (x - h)^2. \tag{14}$$

If h is positive the curve is translated to the right, as shown in Figure 18b. If h is negative the curve is translated to the left. The quadratic polynomial in (14) has leading term x^2. To obtain a leading term ax^2 we multiply the right member of (14) by a. This gives us the quadratic polynomial

$$y = a(x - h)^2, \tag{15}$$

whose graph is shown in Figure 18c. If a is positive, this represents a change of scale in the vertical direction by the scaling factor a. Now we translate this curve vertically by an amount k to obtain

$$y = a(x - h)^2 + k. \tag{16}$$

If k is positive the curve is translated upward, as shown by the example in Figure 18d, and if k is negative it is translated downward. The right member of Equation (16) is $ax^2 - (2ah)x + (ah^2 + k)$. Now we choose h and k to make this quadratic polynomial equal to $ax^2 + bx + c$. Therefore we want

$$\begin{aligned} ax^2 - (2ah)x + (ah^2 + k) &= \\ ax^2 + \quad bx + c. \end{aligned} \tag{17}$$

The quadratic terms already match. To match the linear terms we need $-2ah = b$, so we take

$$h = -\frac{b}{2a}. \tag{18}$$

To match the constant terms we need $ah^2 + k = c$, so we take $k = c - ah^2$. Using the value h in (18) we find

$$k = c - \frac{b^2}{4a}. \tag{19}$$

The result of this geometric process is the algebraic identity

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}, \tag{20}$$

which states that every quadratic polynomial is a constant times the square of a linear polynomial, plus another constant. The right member of (20) is said to be obtained from the left by *completing the square*.

The last two terms in the right member of (20) can be combined as follows:

$$c - \frac{b^2}{4a} = \frac{4ac - b^2}{4a}.$$

The numerator, $4ac - b^2$, is the negative of the discriminant of the quadratic polynomial, $D = b^2 - 4ac$. Therefore Equation (20) can be rewritten as follows:

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{D}{4a}. \tag{21}$$

Equation (21) unlocks many problems concerning quadratic polynomials. First, it helps us find the zeros of a quadratic polynomial. If we equate the right member to zero and divide by a we obtain

$$(x + \frac{b}{2a})^2 = \frac{D}{4a^2},$$

which, when solved for x, gives the quadratic formula stated earlier in (13):

$$x = \frac{-b \pm \sqrt{D}}{2a}, \quad \text{where } D = b^2 - 4ac. \tag{13}$$

Equation (21) also tells us how to find the vertex of the parabola, the point where it has its minimum (if $a > 0$) or its maximum (if $a < 0$). The coordinates of the vertex are (h, k), where

$$h = -\frac{b}{2a}, \text{ and } k = -\frac{D}{4a}.$$

Thus, we have shown that the graph of the general quadratic $y = ax^2 + bx + c$ can always be obtained from the graph of the special quadratic $y = x^2$ as follows: If the leading coefficient a is positive, translate the graph horizontally by the amount $-b/(2a)$, then change the scale in the vertical direction by the factor a, and then translate vertically by the amount $-D/(4a)$. If the leading coefficient is negative, use the same process, but begin with a reflection through the x axis and replace a by $-a$ throughout.

EXERCISES

1. Find the real zeros of each of the following quadratic polynomials. In each case, draw the graph of the corresponding parabola and find the coordinates of its vertex.

(a) $x^2 - 6x - 7$ (b) $2x^2 - x + 1$ (c) $6x^2 + x - 2$ (d) $9x^2 + 9x - 28$

2. In each case, find all real x for which the quadratic polynomial is positive:

(a) $3x^2 - 6x + 2$ (b) $6x^2 - 7x + 3$ (c) $-2x^2 - 9x + 5$ (d) $-4x^2 - 4x - 2$

3. Find all b such that the two zeros of the quadratic polynomial $4x^2 + bx + 2$ will be equal.

4. The quadratic polynomial $x^2 + 5x + c$ has one zero equal to 4. Determine the other zero.

5. The quadratic polynomial $x^2 + bx + 36$ has one zero equal to -9. Determine the other zero.

6. If the quadratic polynomial $ax^2 + bx + c$ has two zeros s and t, show that their sum $s + t$ is $-b/a$, their product st is c/a, and that $ax^2 + bx + c = a(x - s)(x - t)$.

7. Use Equation (21) to show that if $a > 0$ the quadratic polynomial $y = ax^2 + bx + c$ takes its minimum value when $x = -\ b/(2a)$, but if $a < 0$ the same x gives the maximum value of the polynomial.

8. The graph of the quadratic polynomial $y = 3x^2 + bx + c$ has its vertex at $(1, 1)$. Find b and c.

9. A farmer has a roll of wire fencing material 72 feet long. (a) Find the dimensions of the rectangular pasture of largest area that can be enclosed by this fence. (b) Solve part (a) if a long stone wall forms one edge of the boundary of the pasture.

10. A farmer wishes to enclose a rectangular pasture of area 5,000 square feet adjacent to a long stone wall. Find the dimensions of the pasture that require the least amount of fencing.

11. Among all positive x and y whose sum is a given number S, prove that the product xy is largest when $x = y = S/2$. Use this to show that among all rectangles of given perimeter, the square has the largest area.

12. Among all right triangles with a given hypotenuse of length c, find (in terms of c) the lengths of the legs for which the area of the triangle is largest. (You may use the Pythagorean Theorem: In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.)

13. Find the square of largest area that can be circumscribed about a given square of edge length L. (Each vertex of the inner square must be on an edge of the outer square.)

14. Find the rectangle of largest area that can be inscribed in a semicircle of radius 1 if the lower base is on a diameter.

5. Cubic polynomials

We have seen that the graph of every quadratic polynomial can be obtained from one basic curve, the graph of $y = x^2$, by horizontal or vertical translation, a vertical change of scale, and a possible reflection through the x axis. The situation for cubic curves is similar, except that three basic cubic curves are needed. These curves, shown in Figure 19, have equations $y = x^3$, $y = x^3 + x$, and $y = x^3 - x$; they can be regarded as prototypes of all cubic curves. It can be shown that every cubic curve can be obtained from one of these by horizontal or vertical translation, by horizontal or vertical change of scale, or by reflection through the x axis.

The three prototypes reveal basic properties of all cubic curves. Each graph has two parts, one that bends downward and one that bends upward. The point where the upward and downward parts meet is called a *point of inflection*. At this point there is a change in direction of bending, from upward to downward, or vice versa. Graphs of quadratics do not have inflection points because they do not change their direction of bending. As seen in Figure 19, the shape of the curve near the inflection point is different in the three prototypes.

Unlike parabolas, cubics do not have an axis of symmetry. However, they do have a *point* of symmetry, the point of inflection. Any straight line passing through the point of inflection intersects the cubic curve at two points equidistant from the point of inflection. This property is easily demonstrated in the three prototypes, each of which has its point of inflection at the origin. If a point (x, y) lies on one of these curves, so does the point $(-x, -y)$ symmetrically located about the origin.

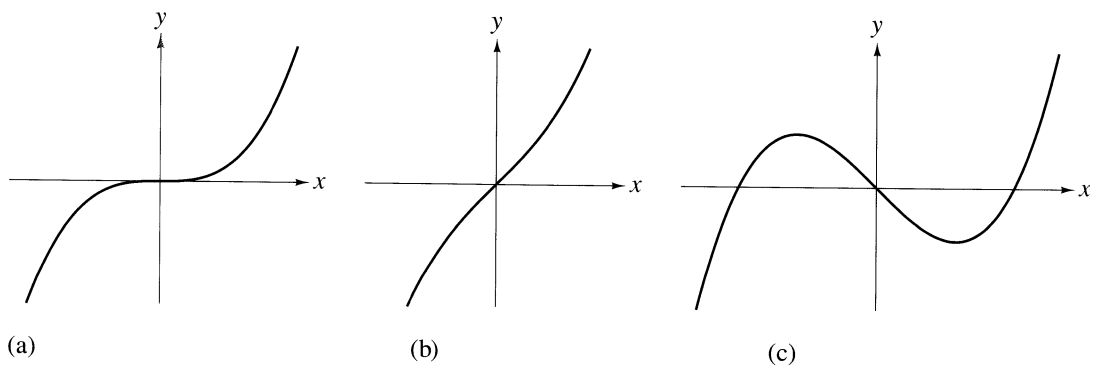


Figure 19. Prototypes of all cubic curves: (a) $y = x^3$; (b) $y = x^3 + x$; (c) $y = x^3 - x$.

The basic shapes of the prototypes do not change if we move them vertically up or down. Each of the graphs in Figures 19a and 19b crosses the x axis at exactly one point, even when the graph is moved up or down. But the graph in Figure 19c crosses the x axis at three distinct points; moving this graph up or down doesn't change its shape but it can change the number of zeros to two or one.

Every cubic curve crosses the x axis at least once. This is because the algebraic sign of a polynomial for large x is governed by its leading term. For example, in the polynomial $f(x) = 2x^3 - 5x^2 - 10x - 75$, the cubic term $2x^3$ overwhelms all the remaining terms if x is large. It is clear that $f(x)$ will be positive if x is large enough, say $x > 1000$. For the same reason, for large negative x, say $x < -1000$, both $2x^3$ and $f(x)$ will be negative. Because $f(x)$ changes sign in the interval between $x = -1000$ and $x = 1000$ there must be at least one point in this interval where $f(x) = 0$. The same kind of reasoning shows that every cubic polynomial changes sign between large negative values of x and large positive values of x, and therefore every cubic polynomial has at least one real zero.

To find the real zeros of a cubic polynomial, we factor the cubic as the product of a linear factor and a quadratic factor. The linear factor has exactly one zero, which provides a zero for the cubic. Additional zeros of the cubic (if any exist) come from the quadratic factor. The number of zeros of any quadratic is 0, 1, or 2, so the number of zeros of a cubic will be 1, 2, or 3.

EXAMPLES. The three prototypes can be split into linear and quadratic factors as follows:

(a) $x^3 = x(x^2)$; (b) $x^3 + x = x(x^2 + 1)$; (c) $x^3 - x = x(x^2 - 1)$.

The linear factor x in each case explains the zero at $x = 0$. In (a) the quadratic factor x^2 also has a double zero at $x = 0$. In (b) the quadratic factor $x^2 + 1$ has no real zeros. In (c) the quadratic factor $x^2 - 1$ can be split into two linear factors, $(x - 1)(x + 1)$, with zeros at 1 and -1, so the cubic has a total of three zeros, one for each linear factor. In (a), where all three zeros are equal, we say the cubic has a triple zero.

Now consider a cubic with three distinct linear factors, say

$$f(x) = (x - a)(x - b)(x - c), \quad \text{where } a < b < c.$$

This cubic has three distinct zeros at $x = a, b, c$, and its graph has the general shape shown in Figure 20. To see why, let's follow the graph from left to right. If $x < a$ all three factors are negative and the graph lies below the x axis. As x approaches a from the left, the factor $x - a$ gets closer to 0, and when x reaches a the graph crosses the x axis. As x increases past a the factor $x - a$ changes from negative to positive, but the other two factors are negative (as long as x is less than b) and the graph lies above the x axis. When x reaches b the graph crosses the x axis again. Then the factor $x - b$ changes sign, so the cubic becomes negative again and its graph stays below the x axis until x reaches c, where the graph crosses the axis a third time. From then on each factor is positive, and as x increases the graph continues to rise forever. Somewhere between a and b the curve has a turning point, or peak, called a *local maximum*. And there is another turning point, or valley, between b and c, called a *local minimum*.

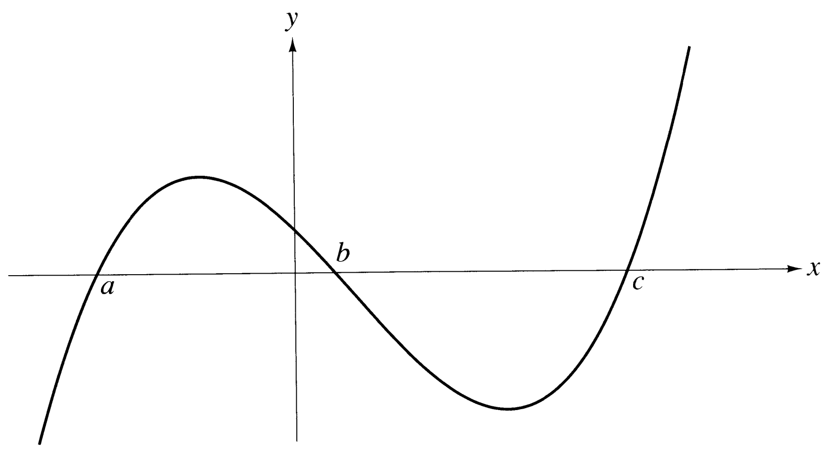


Figure 20. A cubic with three distinct zeros.

When the zeros a and b are brought closer together and eventually allowed to coincide, as in Figure 21a, the cubic touches the x axis and is said to have a *double zero* at its local maximum. In Figure 21b, the zeros b and c are made to coincide, giving a double zero at the local minimum. If all three zeros coincide, the peak and valley come together at a point of inflection, as shown in Figure 19a.

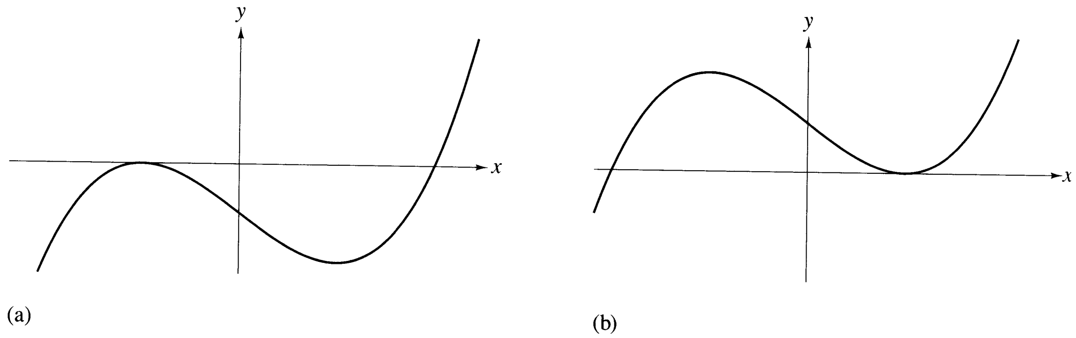


Figure 21. Cubics with a double zero (a) at the local maximum, and (b) at the local miminum.

6. *Polynomials of higher degree*

Figure 22 shows the graph of a polynomial of degree four, $y = \frac{1}{2}x^4 - \frac{5}{2}x^2 + 2$. This polynomial can be factored as a product of a constant factor times four distinct linear factors, each of which corresponds to a zero:

$$\frac{1}{2}x^4 - \frac{5}{2}x^2 + 2 = \frac{1}{2}(x^4 - 5x^2 + 4) = \frac{1}{2}(x^2 - 4)(x^2 - 1) = \frac{1}{2}(x - 2)(x + 2)(x - 1)(x + 1).$$

All polynomials with four distinct real zeros have graphs with common properties. Between two consecutive real zeros there is a valley (local minimum) or a peak (local maximum). And there are two points of inflection where the curve changes its direction of bending.

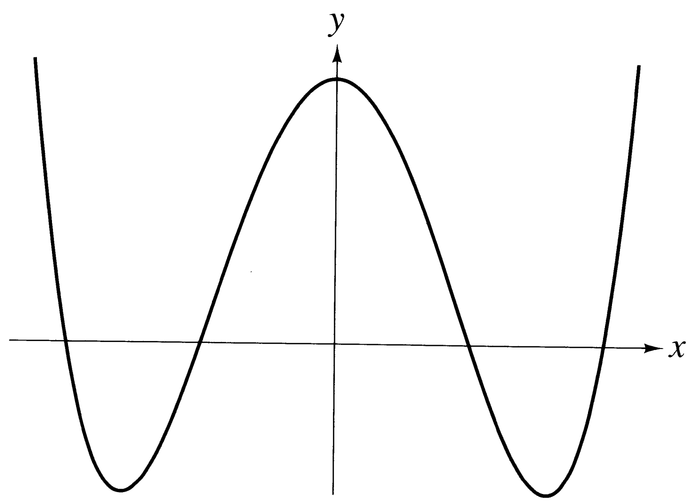


Figure 22. A polynomial of degree four with four distinct real zeros.

Figure 23a shows what happens as two consecutive zeros are made to coincide. In Figure 23b three consecutive zeros coincide, and a peak and valley combine at a point of inflection. In Figure 23c, all four zeros coincide and we get a new valley whose floor is much flatter. Unlike cubic curves, which always cross the x axis, some quartic curves do not cross at all (Figure 23d).

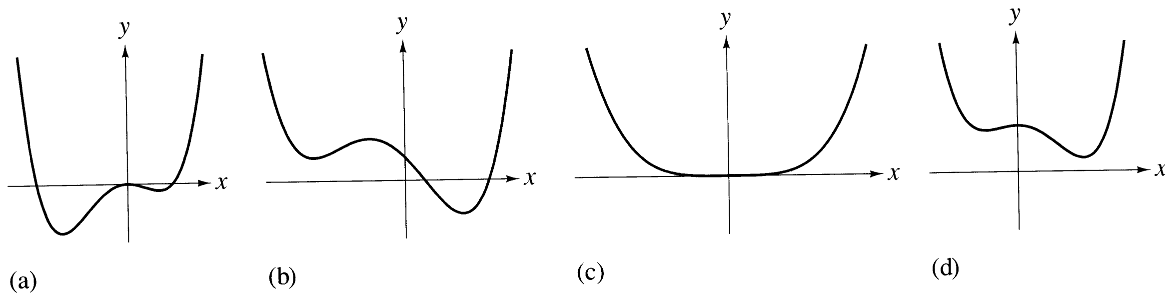


Figure 23. Quartic curves with (a) three real zeros; (b) two real zeros; (c) one real zero; (d) no real zeros.

The number of inflection points of a general quartic curve can be 0 or 2. The curve in Figure 23c has none, but each of those in Figures 23a, 23b and 23d has two.

We have seen that there is one prototype for quadratic curves, and three prototypes for cubic curves. For degree four or higher the number of prototypes is no longer finite. For example, for quartic curves the prototypes consist of the three curves

$$y = x^4, \quad y = x^4 - x^2, \quad y = x^4 + x^2,$$

together with an infinite family of curves given by

$$y = x^4 + x + rx^2, \tag{20}$$

where the coefficient r multiplying the quadratic term runs through all real numbers. It can be shown that every quartic curve can be obtained from one of these by horizontal or vertical translation, by horizontal or vertical change of scale, or by reflection through the x axis. Figure 24 shows some members of the family in (20). (An analysis of the prototypes for polynomial graphs of all degrees has been made in an unpublished article by Nitsa Movshovitz-Hadar and Alla Shmukler from the Technion-Israel Institute of Technology. We are grateful to these authors for providing us with a preprint of their article.)

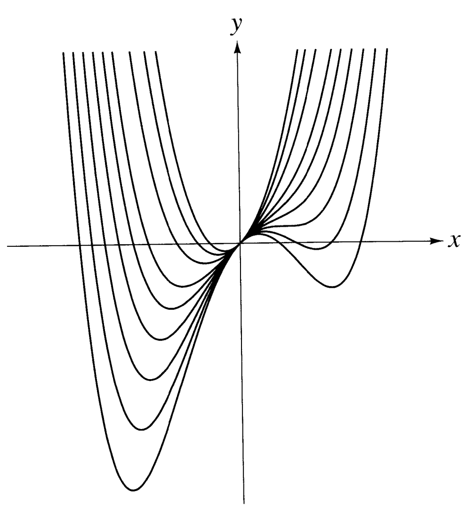


Figure 24. Members of the family in (20).

7. Calculation of polynomials

Polynomials are pleasant to work with because their values can be calculated easily. The only operations involved are multiplication and addition. For example, suppose we wish to compute the value of the quartic polynomial

$$f(x) = 6 - 5x + 3x^2 - 7x^3 + 2x^4$$

when $x = 4$. Substituting $x = 4$ in the formula we find

$$f(4) = 6 - 5{\cdot}4 + 3{\cdot}4^2 - 7{\cdot}4^3 + 2{\cdot}4^4$$

$$= 6 - 20 + 48 - 448 + 512 = 98.$$

This calculation, which starts with the term of lowest degree and works up to the term of highest degree, requires seven multiplications and four additions. (Three multiplications to calculate the powers 4^2, 4^3, and 4^4, plus four more to multiply the powers of 4 by their respective coefficients.)

The calculation can be done another way with fewer multiplications. First write the terms of the polynomial in decreasing order:

(21) $$f(x) = 2x^4 - 7x^3 + 3x^2 - 5x + 6.$$

Start with the leading coefficient and multiply it by x to obtain $2x$. Add the next coefficient, –7, and multiply by x again, to obtain $2x^2 - 7x$. Add the next coefficient, 3, and multiply by x again to obtain $2x^3 - 7x^2 + 3x$. Then add the next coefficient, –5, and multiply by x to obtain $2x^4 - 7x^3 + 3x^2 - 5x$. Finally, add 6 to obtain $f(x)$. This method expresses the polynomial $f(x)$ in the following 'nested' form:

(22) $$f(x) = x\{x[x(2x - 7) + 3] - 5\} + 6.$$

To calculate $f(x)$ this way requires four multiplications and four additions. It is more efficient than the first method and is also easy to program on a calculator. When $x = 4$ the calculation can be arranged in tabular form as follows:

$$\begin{array}{r|rrrrr} & 2 & -7 & 3 & -5 & 6 \\ 4\cdot & & 8 & 4 & 28 & 92 \\ \hline & 2 & 1 & 7 & 23 & 98 \end{array}$$

In this table, the first row contains the coefficients as they appear in (21), where the polynomial is written in decreasing powers of x. The bottom row is obtained by writing down 2, the leading coefficient, which is then multiplied by 4 (the value of x). The product, 8, is written in the second row below the next coefficient, –7, and added to it. This gives 1, the next entry in the last row. This, in turn, is multiplied by 4 to give 4, which is entered in the second row and added to 3 to obtain 7, the next entry in the last row. This is multiplied by 4 to give 28, which is entered in the second row and added to –5 to give the next entry, 23, in the last row. Finally, 23 is multiplied by 4 to give 92, which is entered in the second row and added to 6 to give 98, or $f(4)$.

This method of evaluating polynomials is sometimes called *Horner's method*, after the British mathematician William George Horner who lived in the nineteenth century. It is similar to a method developed a few years earlier by the Italian mathematician Paolo Ruffini. Neither Horner nor Ruffini knew of the work of the other, and apparently neither was aware that essentially the same process had been developed a century earlier by Isaac Newton, and six centuries earlier by the Chinese mathematician Chin Kiu-shao.

The foregoing calculation can be done on a pocket calculator by carrying out the following steps:

Press buttons	***Result***
$4 \times 2 =$	8
$-7 =$	1
$\times 4 =$	4
$+ 3 =$	7
$\times 4 =$	28
$-5 =$	23
$\times 4 =$	92
$+6 =$	98

Some pocket calculators require the sequence of operations to be performed in slightly different form:

Press button	***Operation***	***Result***
2	Enter	2
4	Multiply	8
7	Subtract	1
4	Multiply	4
3	Add	7
4	Multiply	28
5	Subtract	23
4	Multiply	92
6	Add	98

Horner's method can be used not only to evaluate polynomials but also to divide a polynomial by a linear factor. For example, the first four numbers in the last row of the table on page 26 can be used as coefficients of a polynomial $Q(x)$ of degree three:

$$Q(x) = 2x^3 + x^2 + 7x + 23.$$

This polynomial is related to the quartic polynomial $f(x)$ in (21) by the equation

$$f(x) = (x - 4)Q(x) + 98, \tag{23}$$

as can be verified by carrying out the calculation indicated on the right of (23). We interpret this equation as representing a division. The polynomial $f(x)$ is divided by the linear polynomial $(x - 4)$, yielding a quotient $Q(x)$ and a remainder 98. Putting $x = 4$ in this equation we get $f(4) = 98$.

This example illustrates a general property of polynomials. Start with any polynomial $f(x)$ of degree $n > 1$ and choose any real number c. Suppose we can express $f(x)$ in the form

$$f(x) = (x - c)Q(x) + r, \tag{24}$$

where $Q(x)$ is a polynomial and r is a constant. Then $Q(x)$, which has degree $n - 1$, is called the *quotient* obtained by dividing $f(x)$ by $(x - c)$, and the constant r is called the *remainder*. Taking $x = c$ in Equation (24) we see that $f(c) = r$. In other words, the remainder r is the value of $f(x)$ when $x = c$. Therefore, Equation (24) can also be written as follows:

$$f(x) = (x - c)Q(x) + f(c). \tag{25}$$

This equation is called *the remainder theorem.* It tells us that if we divide a polynomial by the linear polynomial $(x - c)$ and obtain a constant remainder, then the remainder is equal to $f(c)$. In particular, if $f(c) = 0$, Equation (25) states that

$$f(x) = (x - c)Q(x).$$

In other words, if c is a zero of $f(x)$, then $(x - c)$ is a factor of $f(x)$. This result is called *the factor theorem.* The ceofficients of the quotient $Q(x)$ can be determined by Horner's method. When used for this purpose, Horner's method is also known as *synthetic division.*

EXAMPLE. Find the quotient and remainder obtained by dividing the polynomial

$$f(x) = x^3 + 6x^2 + 11x + 6$$

by each of the linear factors $(x - 1)$ and $(x + 1)$.

Solution. We calculate each of $f(1)$ and $f(-1)$ by Horner's method and we obtain:

$$\begin{array}{r|rrrr} & 1 & 6 & 11 & 6 \\ 1\cdot & & 1 & 7 & 18 \\ \hline & 1 & 7 & 18 & 24 \end{array} \qquad\qquad \begin{array}{r|rrrr} & 1 & 6 & 11 & 6 \\ -1\cdot & & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

The first calculation shows that

$$f(x) = (x - 1)(x^2 + 7x + 18) + 24,$$

so the remainder obtained by dividing $f(x)$ by $(x - 1)$ is 24. The second calculation shows that

$$f(x) = (x + 1)(x^2 + 5x + 6),$$

with 0 remainder. Hence $(x + 1)$ is a factor of $f(x)$ and -1 is a zero of $f(x)$. Note that the quadratic factor can be split into two linear factors, $(x + 2)(x + 3)$, so $f(x)$ is the product of three linear factors,

$$f(x) = (x + 1)(x + 2)(x + 3),$$

from which we see that the zeros of $f(x)$ are -1, -2, and -3.

EXERCISES

1. Starting with the number 4, the number 4^5 can be calculated with four multiplications:

$$4 \times 4 = 16, \quad 4 \times 16 = 64, \quad 4 \times 64 = 256, \quad 4 \times 256 = 1024.$$

The number of multiplications can be reduced to three: $4 \times 4 = 16$, $16 \times 16 = 256$, $4 \times 256 = 1024$. Find the smallest number of multiplications necessary to calculate each of 4^6, 4^7, 4^8, 4^9, and 4^{10}.

2. Find the quotient and the remainder obtained by dividing each of the following polynomials by the linear factor indicated. Note that missing powers of x have 0 as their coefficient.

(a) $5x^5 - 7x^3 + 6x^2 - 2x + 4$ by $x - 1$. (b) $x^4 + 3x^3 - 2x + 7$ by $x + 3$.

(c) $3x^6 - 7x^5 + 5x^4 - x^2 - 6x - 8$ by $x + 2$. (d) $5x^6 - 6x^4 + 1$ by $x + 1$.

(e) $x^4 + x^3 - x^2 + 1$ by $3x + 2$. Hint: $3x + 2 = 3(x + \frac{2}{3})$.

3. Find all constants c such that $x^3 + 6x^2 + 4x + c$ is divisible by (a) $x + 1$; (b) $x + 2$; (c) $x + c$.

4. (a) Show that $x - c$ is a factor of $x^n - c^n$ for every positive integer n, and determine the quotient.

(b) Show that $x + c$ is a factor of $x^n + c^n$ for every positive odd integer n.

(c) If n is even, find the remainder when $x^n + c^n$ is divided by $x + c$.

(d) Show that $x + c$ is a factor of $x^n - c^n$ for every positive even integer n.

5. When the polynomial $2x^4 - 6x^2 + 1$ is divided by $x + 2$ the quotient is $Q_3(x)$, a polynomial of degree 3, and the remainder is 9. Determine the polynomial $Q_3(x)$. Divide $Q_3(x)$ by $x + 2$ to obtain a quotient $Q_2(x)$ of degree 2 and remainder -40. Divide $Q_2(x)$ by $x + 2$ to obtain a quotient $Q_1(x)$ of degree 1 and remainder 42. Divide $Q_1(x)$ by $x + 2$ to obtain a quotient $Q_0(x) = 2$ and remainder -16. Show that, as a result of these calculations, the given polynomial can be expressed in powers of $(x + 2)$:

$$2x^4 - 6x^2 + 1 = 9 - 40(x + 2) + 42(x + 2)^2 - 16(x + 2)^3 + 2(x + 2)^4,$$

the respective coefficients 9, -40, 42, -16, and 2 being the remainders and the final quotient obtained by this process. This example illustrates a general property: Every polynomial can be expressed in powers of $(x - c)$ for any real number c. The coefficients of the respective powers of $(x - c)$ can be determined conveniently by repeated application of synthetic division as was done in the example. In fact, Horner presented his method as an efficient process for determining these coefficients.

6. Special polynomials, called *factorial polynomials*, are defined as follows:

$$x^{(0)} = 1, \quad x^{(1)} = x, \quad x^{(2)} = x(x - 1), \quad x^{(3)} = x(x - 1)(x - 2), \quad x^{(4)} = x(x - 1)(x - 2)(x - 3), \quad \text{etc.}$$

(a) Show that $(x + 1)^{(3)} - x^{(3)} = 3x^{(2)}$, and $(x + 1)^{(4)} - x^{(4)} = 4x^{(3)}$.

(b) Use (a) to deduce that $1^{(2)} + 2^{(2)} + \cdots + n^{(2)} = (n + 1)^{(3)}/3$ and $1^{(3)} + 2^{(3)} + \cdots + n^{(3)} = (n + 1)^{(4)}/4$.

(c) Show that $x^3 = x^{(1)} + 3x^{(2)} + x^{(3)}$ and deduce that

$$1^3 + 2^3 + \cdots + n^3 = [n(n + 1)/2]^2.$$

8. *Recap*

The shapes of polynomial graphs can be summarized in a visual catalog arranged according to degree, as illustrated by the examples on pages 4 and 5 of this booklet.

Degree zero. The graphs are straight lines of slope zero, parallel to the x axis.

Degree one. The graphs are straight lines with nonzero slope. The leading coefficient is equal to the slope, and there is exactly one real zero.

Degree two. The graphs are parabolas. There is one basic shape, or prototype, the graph of $y = x^2$, and all others come from this by a combination of horizontal or vertical translation, vertical scaling, or reflection through the x axis. Each parabola has an axis of symmetry intersecting the curve at its vertex. If the leading coefficient is positive the parabola opens upward with a valley at the vertex. If the leading coefficient is negative, it opens downward with a peak at the vertex. Quadratic polynomials can have two distinct real zeros, one double zero, or no real zero.

Degree three. There are three prototypes, $y = x^3$, $y = x^3 + x$, and $y = x^3 - x$. Each cubic curve has a point of inflection which is also a point of symmetry. The number of real zeros can be one, two, or three. There can be exactly one peak (local maximum) and exactly one valley (local minimum), or none.

Degree four. The prototypes consist of the three curves $y = x^4$ $y = x^4 - x^2$, $y = x^4 + x^2$, together with an infinite family of curves given by $y = x^4 + x + rx^2$, where r takes all real values. The number of real zeros is at most four. The graph has at least one valley or one peak. There can be two valleys and one peak, or two peaks and one valley, or just one peak and no valley, or one valley and no peak.

Higher degree. As the degree increases, the number of prototypes also increases. The maximum number of real zeros is equal to the degree, and the number of peaks and valleys is at most one less than the degree. Polynomials of odd degree always have at least one real zero.

Polynomials are easy to calculate numerically because only a finite number of multiplications and additions are required. The function values can be calculated efficiently by Horner's method. Many curves that are not graphs of polynomials can be approximated with great accuracy by polynomial graphs. For this reason, polynomial approximations are often used in designing hand calculators. When you press a special function key such as a sine, cosine, logarithm, or exponential function, the program inside the calculator often computes a polynomial approximation that gives the accuracy required. Figure 25 shows the graphs of polynomials of degree 5, 7, 11, and 13 closely approximating a portion of a sine curve.

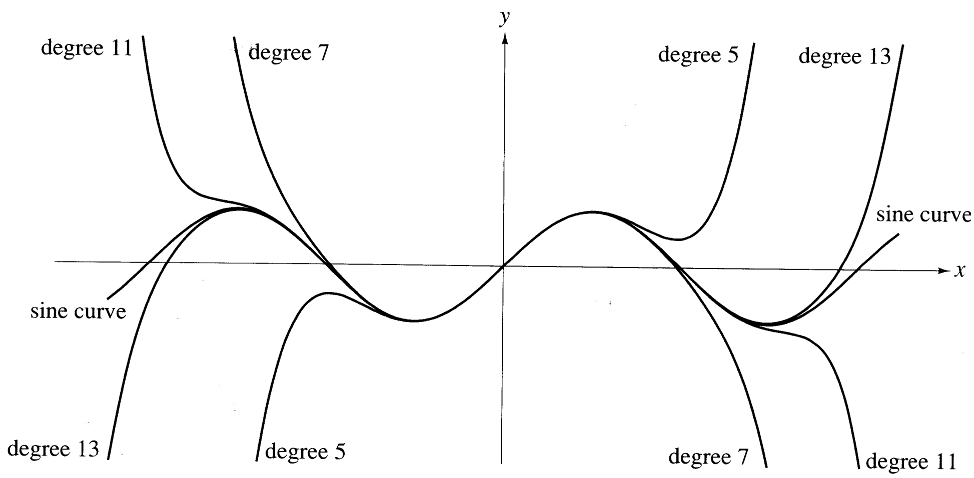


Figure 25. Graphs of polynomials approximating a portion of a sine curve.

INDEX

Project MATHEMATICS!

COMPUTER ANIMATED MATHEMATICS VIDEOTAPES

Polynomials is part of a series of modules that use computer animation to help instructors teach basic concepts in mathematics. Each module consists of a videotape, about 20 minutes in length, and a workbook to guide the students through the video, elaborating on the important ideas. The modules are used as support material for existing courses in high school and community college classrooms, and may be copied without charge for educational use.

Based at the California Institute of Technology, *Project MATHEMATICS!* has attracted as partners the departments of education of 32 states in a consortium whose members reproduce and distribute the videotapes and written materials to public schools. The project is headed by Tom M. Apostol, professor of mathematics at Caltech and an internationally known author of mathematics textbooks. Co-director of the project is James F. Blinn, one of the world's leading computer animators, who is well known for his Voyager planetary flyby simulations. Blinn and Apostol worked together previously as members of the academic team that produced *The Mechanical Universe*, an award-winning physics course for television.

The first videotape produced by *Project MATHEMATICS!*, entitled *The Theorem of Pythagoras,* has been distributed to thousands of classrooms nationwide. It received first-place awards at many international competitions: a Gold Medal at the 1988 International Film & TV Festival of New York; a Gold Apple Award at the 1989 National Educational Film & Video Festival, Chicago, Illinois; an Electra Certificate, Best of Category, at the 1989 Birmingham International Educational Film Festival; and a Gold Cindy at the 1989 Cindy Competition, Los Angeles, California. *The Story of Pi* was awarded a Gold Apple Award at the 1990 National Educational Film & Video Festival in Oakland, and a Red Ribbon Award at the 1990 American Film and Video Festival.

Information about the project can be obtained by writing to the project director at the address on the title page of this booklet. Copies of the videotape and workbook on *The Theorem of Pythagoras, The Story of Pi, Similarity*, or *Polynomials* can be obtained at nominal charge from the Caltech Bookstore, 1-51 Caltech, Pasadena, CA 91125. (Tel: 818-356-6161) The first two titles can also be obtained from the National Council of Teachers of Mathematics, and the first three from the Mathematical Association of America at the addresses given below.

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